

ON LOCAL CONVEXITY OF NONLINEAR MAPPINGS BETWEEN BANACH SPACES

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ABSTRACT. We find conditions for a smooth nonlinear map $f : U \rightarrow V$ between open subsets of Hilbert or Banach spaces to be locally convex in the sense that for some c and each positive $\varepsilon < c$ the image $f(B_\varepsilon(x))$ of each ε -ball $B_\varepsilon(x) \subset U$ is convex. We give a lower bound on c via the second order Lipschitz constant $\text{Lip}_2(f)$, the Lipschitz-open constant $\text{Lip}_o(f)$ of f , and the 2-convexity number $\text{conv}_2(X)$ of the Banach space X .

INTRODUCTION

The local convexity of nonlinear mappings of Banach spaces is important in many branches of applied mathematics [1, 2, 13, 18, 19, 20, 21], in particular, in the theory of nonlinear differential-operator equations, optimization and control theory etc. Locally convex maps appear naturally in various problems of Fixed Point Theory [7, 8, 9] and Nonlinear Analysis [12, 16, 17, 22].

Let X, Y be Banach spaces. A map $f : U \rightarrow Y$ defined on an open subset $U \subset X$ is called *locally convex* at a point $x \in U$ if there is a positive constant $c > 0$ such that for each positive $\varepsilon \leq c$ and each point $x \in U$ with $B_\varepsilon(x) \subset U$ the image $f(B_\varepsilon(x))$ is convex. Here $B_\varepsilon(x) = \{y \in X : \|x - y\| < \varepsilon\}$ stands for the open ε -ball centered at x . The local convexity of f at x can be expressed via the *local convexity radius*

$$\text{lcr}_x(f) = \sup \{c \in [0, +\infty) : \forall \varepsilon \leq c \forall x \in U \text{ with } B_\varepsilon(x) \subset U \text{ the set } f(B_\varepsilon(x)) \text{ is convex}\}.$$

It follows that f is locally convex at $x \in U$ if and only if $\text{lcr}_x(f) > 0$.

A map $f : U \rightarrow Y$ is defined to be

- *locally convex* if f is locally convex at each point $x \in U$;
- *uniformly locally convex* if its *local convexity radius* $\text{lcr}(f) = \inf_{x \in U} \text{lcr}_x(f)$ is not equal to zero.

For example, if a homeomorphism $f : U \rightarrow V$ between open subsets $U \subset X$, $V \subset Y$ with $f(0) = 0 \in U$ is norm convex in the sense that

$$\left\| f\left(\frac{x+x'}{2}\right) \right\| \leq \frac{1}{2}(\|f(x)\| + \|f(x')\|) \quad \text{for all } x, x' \in f(U),$$

then the inverse map f^{-1} is locally convex at the point $y = 0$. In particular, if Y is a Banach lattice with the order \leq and a homeomorphism $f : U \rightarrow V$ is Jensen convex, i.e.

$$f\left(\frac{x+x'}{2}\right) \leq \frac{1}{2}(f(x) + f(x'))$$

for all $x, x' \in U$, then the inverse map f^{-1} is locally convex at the point $y = 0$.

In this paper we find some conditions on a map $f : U \rightarrow Y$ guaranteeing that f uniformly locally convex, and give a lower bound on the local convexity radius $\text{lcr}(f)$ of f . This bound depends on the second order Lipschitz constant $\text{Lip}_2(f)$ of f , the Lipschitz-open constant $\text{Lip}_o(f)$ of f , and the 2-convexity number $\text{conv}_2(X)$ of the Banach space X .

1. BANACH SPACES WITH MODULUS OF CONVEXITY OF POWER TYPE 2

The *modulus of convexity* of a Banach space X is the function $\delta_X : [0, 2] \rightarrow [0, 1]$ assigning to each number $t \geq 0$ the real number

$$\delta_X(t) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in S_X, \|x - y\| \geq t \right\},$$

Key words and phrases. Locally convex mapping, Hilbert and Banach spaces, modulus of convexity, modulus of smoothness, Lipschitz-open maps.

where $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of the Banach space X . By [15, p.60], the modulus of convexity can be equivalently defined as

$$\delta_X(t) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \|x - y\| \geq t \right\},$$

where $B_X = \{x \in X : \|x\| \leq 1\}$ is the closed unit ball of X .

Any Hilbert space E of dimension $\dim(E) > 1$ has modulus of convexity

$$\frac{1}{8}t^2 \leq \delta_E(t) = 1 - \sqrt{1 - (t/2)^2} \leq \frac{1}{4}t^2.$$

By [15, p.63] or [10], $\delta_X(t) \leq \delta_E(t) \leq \frac{1}{4}t^2$ for each Banach space X .

Following [15, p.63], [5, p.154], we say that the Banach space X has *modulus of convexity of power type p* if there is a constant $L > 0$ such that $\delta_X(t) \geq L \cdot t^p$ for all $t \in [0, 2]$. It follows from $L t^p \leq \delta_X(t) \leq \frac{1}{8}t^2$ that $p \geq 2$. Hilbert spaces have modulus of convexity of power type 2. Many examples of Banach spaces with modulus of convexity of power type 2 can be found in [15, §1.e], [5, Ch.V], [3], [14], and [11]. In particular, the class of Banach spaces with modulus of convexity of power type 2 includes the Banach spaces L_p for $1 < p \leq 2$, and reflexive subspaces of the Banach space L_1 . By [10], a Banach space X has modulus of convexity of power type 2 if and only if for any sequences $(x_n)_{n \in \omega}$ and $(y_n)_{n \in \omega}$ in X the convergence $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 \rightarrow 0$ implies $\|x_n - y_n\| \rightarrow 0$.

For a Banach space X consider the constant

$$\text{conv}_2(X) = \inf \left\{ \frac{1 - \left\| \frac{x+y}{2} \right\|}{\|x - y\|^2} : x, y \in B_X, x \neq y \right\} \geq 0$$

called the *2-convexity number* of X and observe that $\text{conv}_2(X) > 0$ if and only if X has modulus of convexity of power type 2. It follows from [15, p.63] or [10] that

$$0 \leq \text{conv}_2(X) \leq \text{conv}_2(\ell_2) = \frac{1}{8}$$

for each Banach space X .

2. MODULI OF SMOOTHNESS OF MAPS OF BANACH SPACES

In this section we recall known information [6, §2.7] on the moduli of smoothness $\omega_n(f, t)$ of a function $f : U \rightarrow Y$ defined on a subset $U \subset X$ of a Banach space X with values in a Banach space Y .

The *n -th modulus of smoothness* of f is defined as

$$\omega_n(f, t) = \sup \{ \|\Delta_h^n(f, x)\| : h \in X, \|h\| \leq t, [x, x + nh] \subset U \}$$

where

$$\Delta_h^n(f, x) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(x + kh)$$

is the n -th difference of f .

In particular,

$$\begin{aligned} \omega_1(f, t) &= \sup \{ \|f(x + h) - f(x)\| : \|h\| \leq t, [x, x + h] \subset U \} \text{ and} \\ \omega_2(f, t) &= \sup \{ \|f(x + h) - 2f(x) + f(x - h)\| : \|h\| \leq t, [x - h, x + h] \subset U \}. \end{aligned}$$

Here $[x, y] = \{tx + (1 - t)y : t \in [0, 1]\}$ stands for the segment connecting two points $x, y \in X$.

The constants

$$\text{Lip}_1(f) = \sup_{t>0} \frac{\omega_1(f, t)}{t} \quad \text{and} \quad \text{Lip}_2(f) = \sup_{t>0} \frac{\omega_2(f, t)}{t^2}$$

are called the *Lipschitz constant* and the *second order Lipschitz constant* of f , respectively.

A function $f : U \rightarrow Y$ is called (*second order*) *Lipschitz* if its (second order) Lipschitz constant $\text{Lip}_1(f)$ (resp. $\text{Lip}_2(f)$) is finite. The second order Lipschitz property of a weakly Gâteaux differentiable function f can be deduced from the Lipschitz property of its derivative f' .

Let us recall [4, p.154] that a function $f : U \rightarrow Y$ is *weakly Gâteaux differentiable* at a point $x \in U$ if there is a bounded linear operator $f'_x : X \rightarrow Y$ (called the *derivative* of f at x) such that for each $h \in X$ and each linear continuous functional $y^* \in Y^*$ we get

$$\lim_{t \rightarrow 0} \frac{y^*(f(x+th) - f(x))}{t} = y^* \circ f'_x(h).$$

If

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - f'_x(h)\|}{\|h\|} = 0,$$

then f is *Fréchet differentiable* at x .

The derivative f'_x belongs to the Banach space $L(X, Y)$ of all bounded linear operators from X to Y , endowed with the operator norm $\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$.

The following two propositions are known and we present their short proofs for completeness.

Proposition 2.1. *Let X, Y be Banach spaces and $U \subset X$ be an open subset. A function $f : U \rightarrow Y$ is Lipschitz if f is weakly Gâteaux differentiable at each point of U and the derivative map $f' : U \rightarrow L(X, Y)$, $f' : x \mapsto f'_x$, is bounded. In this case $\text{Lip}_1(f) \leq \|f'\|_\infty = \sup_{x \in U} \|f'_x\|$.*

Proof. Let $L = \|f'\|_\infty$. The inequality $\text{Lip}_1(f) \leq L = \|f'\|_\infty$ will follow as soon as we check that

$$\|f(x+h) - f(x)\| \leq L\|h\|$$

for any $x \in U$ and $h \in X$ with $[x, x+h] \subset U$. Using the Hahn-Banach Theorem, find a linear continuous functional $y^* \in Y^*$ with unit norm $\|y^*\| = 1$ such that $y^*(f(x+h) - f(x)) = \|f(x+h) - f(x)\|$. The weak Gâteaux differentiability of f implies that the function

$$g : [0, 1] \rightarrow \mathbb{C}, \quad g : t \mapsto y^*(f(x+th) - f(x))$$

is differentiable and $g'(t) = y^* \circ f'_{x+th}(h)$ for each $t \in [0, 1]$. Then

$$\|g'\|_\infty \leq \|y^*\| \cdot \|f'_{x+th}\| \cdot \|h\| \leq 1 \cdot \|f'\|_\infty \cdot \|h\| = L \cdot \|h\|$$

and

$$\|f(x+h) - f(x)\| = |g(1) - g(0)| = \left| \int_0^1 g'(t) dt \right| \leq \int_0^1 |g'(t)| dt \leq L\|h\| \int_0^1 dt = L\|h\|.$$

□

Proposition 2.2. *Let X, Y be Banach spaces and $U \subset X$ be an open subset. Assume that a function $f : U \rightarrow Y$ is weakly Gâteaux differentiable at each point of U and the derivative map $f' : U \rightarrow L(X, Y)$, $f' : x \mapsto f'_x$, is Lipschitz. Then*

- (1) f is Fréchet differentiable at each point of U ;
- (2) f is second order Lipschitz with $\text{Lip}_2(f) \leq \text{Lip}_1(f')$.

Proof. Let $L = \text{Lip}_1(f')$. The Fréchet differentiability of f at a point $x \in U$ will follow as soon as we check that

$$\|f(x+h) - f(x) - f'_x(h)\| \leq \frac{1}{2}L\|h\|^2$$

for each $h \in X$ with $[x, x+h] \subset U$. Using the Hahn-Banach Theorem, choose a linear continuous functional $y^* \in Y^*$ such that $\|y^*\| = 1$ and $y^*(f(x+h) - f(x) - f'_x(h)) = \|f(x+h) - f(x) - f'_x(h)\|$. The weak Gâteaux differentiability of f implies that the function

$$g : [0, 1] \rightarrow \mathbb{C}, \quad g : t \mapsto y^*(f(x+th) - f(x) - tf'_x(h)),$$

is differentiable. Moreover, for each $t \in [0, 1]$ we get $g'(t) = y^* \circ f'_{x+th}(h) - y^* \circ f'_x(h)$ and

$$|g'(t)| = |y^*(f'_{x+th}(h) - f'_x(h))| \leq \|y^*\| \cdot \|f'_{x+th}(h) - f'_x(h)\| \leq \|f'_{x+th} - f'_x\| \cdot \|h\| \leq \text{Lip}_1(f') \cdot \|th\| \cdot \|h\| = tL\|h\|^2.$$

Then

$$\|f(x+h) - f(x) - f'_x(h)\| = |g(1) - g(0)| = \left| \int_0^1 g'(t) dt \right| \leq \int_0^1 |g'(t)| dt \leq \int_0^1 tL\|h\|^2 dt = \frac{1}{2}L\|h\|^2.$$

To see that f is second order Lipschitz, observe that for each $h \in X$ with $[x - h, x + h] \subset U$ we get

$$\begin{aligned} \|f(x + h) - 2f(x) + f(x - h)\| &= \|f(x + h) - f(x) - f'_x(h) + f(x - h) - f(x) - f'_x(-h)\| \leq \\ &\leq \|f(x + h) - f(x) - f'_x(h)\| + \|f(x - h) - f(x) - f'_x(-h)\| \leq 2\frac{1}{2}L\|h\|^2 = L\|h\|^2, \end{aligned}$$

which implies that $\text{Lip}_2(f) \leq L = \text{Lip}_1(f')$. \square

3. LIPSCHITZ-OPEN MAPS

Let X, Y be Banach spaces. A map $f : U \rightarrow Y$ defined on an open subset $U \subset X$ is called *Lipschitz-open* if there is a positive constant c such that for each $x \in X$ and $\varepsilon > 0$ with $B_\varepsilon(x) \subset U$ we get $B_{c\varepsilon}(f(x)) \subset f(B_\varepsilon(x))$. Observe that a map $f : U \rightarrow Y$ is Lipschitz-open if and only if its *Lipschitz-open constant*

$$\text{Lip}_o(f) = \sup \{c \in [0, \infty) : \forall x \in U \forall \varepsilon > 0 \ B_\varepsilon(x) \subset U \Rightarrow B_{c\varepsilon}(f(x)) \subset f(B_\varepsilon(x))\}$$

is strictly positive.

A map $f : U \rightarrow Y$ is *locally Lipschitz-open* if each point $x \in U$ has an open neighborhood $W \subset U$ such that the restriction $f|_W : W \rightarrow Y$ is Lipschitz-open.

Observe that a bijective map $f : X \rightarrow Y$ between Banach spaces is Lipschitz-open if and only if the inverse map $f^{-1} : Y \rightarrow X$ is Lipschitz. In this case $\text{Lip}_o(f) = \text{Lip}_1(f^{-1})$.

The following proposition can be derived from Theorem 15.5 of [4].

Proposition 3.1. *Let X, Y be Banach spaces. A map $f : U \rightarrow Y$ defined on an open subspace U of X is locally Lipschitz-open if*

- (1) *f is weakly Gâteaux differentiable and the derivative $f'_x : X \rightarrow Y$ is surjective at each point $x \in U$;*
- (2) *the derivative $f' : U \rightarrow L(X, Y)$ is Lipschitz.*

4. MAIN RESULTS

Theorem 4.1. *Let X, Y be Banach spaces. A map $f : U \rightarrow Y$ defined on an open subspace $U \subset X$ is uniformly locally convex if*

- (1) *the Banach space X has modulus of convexity of power type 2,*
- (2) *f is second order Lipschitz;*
- (3) *f is Lipschitz-open.*

Moreover, in this case f has local convexity radius $\text{lcr}(f) \geq 8 \cdot \text{Lip}_o(f) \cdot \text{conv}_2(X) / \text{Lip}_2(f) > 0$.

Proof. Given any point $x_0 \in U$ and a positive $\varepsilon \leq 8 \cdot \text{Lip}_o(f) \cdot \text{conv}_2(X) / \text{Lip}_2(f)$ with $B_\varepsilon(x_0) \subset U$, we need to prove that the image $f(B_\varepsilon(x_0))$ is convex. Without loss of generality, $x_0 = 0$.

Claim 4.2. *For any points $a, b \in f(B_\varepsilon(x_0))$ we get $(a + b)/2 \in f(B_\varepsilon(x_0))$.*

Proof. Find two points $x, y \in B_\varepsilon(x_0) = B_\varepsilon(0)$ with $a = f(x)$ and $b = f(y)$, and consider the midpoint $z = (x + y)/2$. Observe that the points $x_\varepsilon = x/\varepsilon$, $y_\varepsilon = y/\varepsilon$, and $z_\varepsilon = z/\varepsilon$ have norms ≤ 1 .

The definition of the 2-convexity number $\text{conv}_2(X)$ guarantees that

$$1 - \frac{1}{\varepsilon}\|z\| = 1 - \|z_\varepsilon\| \geq \text{conv}_2(X) \cdot \|x_\varepsilon - y_\varepsilon\|^2 = \frac{1}{\varepsilon^2} \text{conv}_2(X) \|x - y\|^2$$

and thus

$$\varepsilon - \|z\| \geq \frac{1}{\varepsilon} \text{conv}_2(X) \|x - y\|^2.$$

Then $B_\delta(z) \subset B_\varepsilon(x_0)$, where

$$\delta = \frac{1}{\varepsilon} \text{conv}_2(X) \|x - y\|^2 \geq \frac{\text{Lip}_2(f)}{8\text{Lip}_o(f) \cdot \text{conv}_2(X)} \text{conv}_2(X) \|x - y\|^2 = \frac{\text{Lip}_2(f)}{8\text{Lip}_o(f)} \|x - y\|^2$$

and hence

$$f(B_\varepsilon(x_0)) \supset f(B_\delta(z)) \supset B_{\text{Lip}_o(f)\delta}(f(z)) = B_\eta(f(z))$$

where $\eta = \text{Lip}_o(f) \delta = \frac{1}{8} \text{Lip}_2(f) \|x - y\|^2$.

The definition of the constant $\text{Lip}_2(f)$ implies that for $h = z - x$, we get

$$\begin{aligned} \|(a+b)/2 - f(z)\| &= \|(f(x) + f(y))/2 - f(z)\| = \frac{1}{2}\|f(z-h) - 2f(z) + f(z+h)\| \leq \\ &\leq \frac{1}{2}\text{Lip}_2(f)\|h\|^2 = \frac{1}{8}\text{Lip}_2(f)\|x-y\|^2 = \eta \end{aligned}$$

and hence $(a+b)/2 \in B_\eta(f(z)) \subset f(B_\varepsilon(x_0))$. \square

Claim 4.2 implies that the closure $\text{cl}_Y(f(B_\varepsilon(x_0)))$ is convex. The Lipschitz-openness of the map f implies that for any numbers $\delta < \eta < \varepsilon$ we get $\text{cl}_Y(f(B_\delta(x_0))) \subset f(B_\eta(x_0))$. Then the open set $f(B_\varepsilon(x_0))$ is convex, being the union

$$f(B_\varepsilon(x_0)) = f\left(\bigcup_{0 < \delta < \varepsilon} B_\delta(x_0)\right) = \bigcup_{0 < \delta < \varepsilon} \text{cl}_Y(f(B_\delta(x_0)))$$

of a linearly ordered chain of convex sets. \square

Taking into account that each Hilbert space X has 2-convexity number $\text{conv}_2(E) \geq \frac{1}{8}$, and applying Theorem 4.1, we get:

Corollary 4.3. *Let Y be a Banach space and U be an open subspace of a Hilbert space X . Each Lipschitz-open second order Lipschitz map $f : U \rightarrow Y$ is uniformly locally convex and has local convexity radius $\text{lcr}(f) \geq \text{Lip}_o(f)/\text{Lip}_2(f) > 0$.*

Theorem 4.1 combined with Propositions 2.2 and 3.1 implies the following two corollaries.

Corollary 4.4. *Let X, Y be Banach spaces. A map $f : U \rightarrow Y$ defined on an open subspace $U \subset X$ is uniformly locally convex if*

- (1) *the Banach space X has modulus of convexity of power type 2,*
- (2) *f is weakly Gâteaux differentiable and the derivative $f' : U \rightarrow L(X, Y)$ is Lipschitz;*
- (3) *f is Lipschitz-open.*

Corollary 4.5. *Let X, Y be Banach spaces. A map $f : U \rightarrow Y$ defined on an open subspace $U \subset X$ is locally convex if*

- (1) *the Banach space X has modulus of convexity of power type 2,*
- (2) *f is weakly Gâteaux differentiable and the derivative $f' : U \rightarrow L(X, Y)$ is Lipschitz;*
- (3) *for each $x \in U$ the derivative $f'_x : X \rightarrow Y$ is surjective.*

5. AN OPEN PROBLEM

We do not know if the requirement on the convexity modulus of the Banach space X is essential in Theorem 4.1 and Corollaries 4.4, 4.5.

Problem 5.1. *Assume that X is a Banach space such that any Lipschitz-open second order Lipschitz map $f : U \rightarrow X$ defined on an open subset $U \subset X$ is locally convex. Has X the modulus of convexity of power type 2? Is X (super)reflexive?*

6. ACKNOWLEDGEMENTS

The fourth author (A.P.) is grateful to Prof. L. Górniewicz for invitation to take part in the VI Symposium of Nonlinear Analysis held 7 – 9 September 2011 in Toruń and for very fruitful discussion and remarks. He also sincerely thanks Prof. A. Augustynowicz for valuable comments on the first draft of the papers and mentioning the references related with the topic studied in the article. Special acknowledgment belongs to the Scientific and Technological Research Council of Turkey (TUBITAK/NASU-111T558) for a partial support of A.K. Prykarpatsky's research.

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